CSE276C - Subspace Methods

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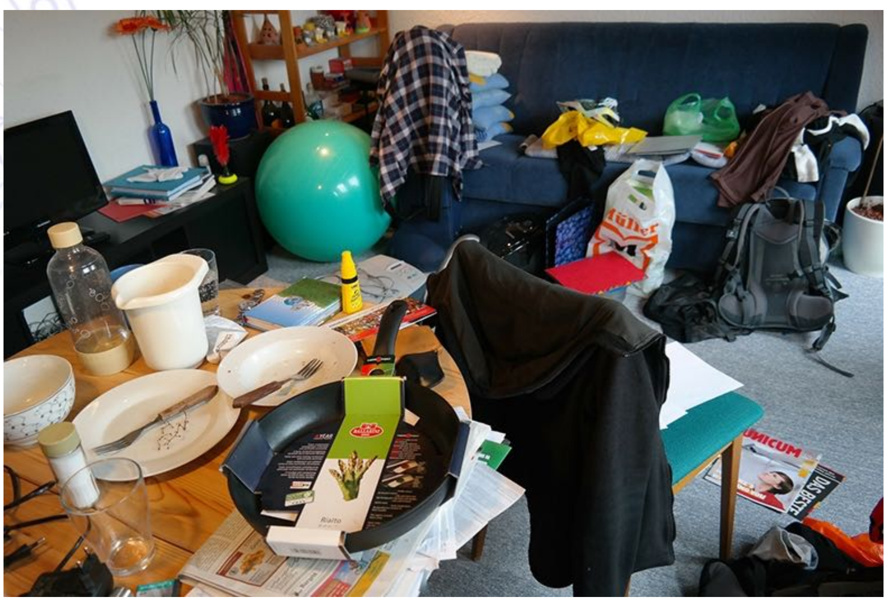
Literature

- Leonardis, A. and Bischof, H., 2000. "Robust recognition using eigenimages". **Computer Vision and Image Understanding**, 78(1), pp.99-118.
- Largely adopted from ECCV tutorial by Leonardis and Bischof

Outline

Recognition of objects in clutter



Recognition of objects in clutter

Typical tasks

- Where can I find a can of coke?
- Check the stove – is it off?
- Put away the groceries in the pantry?

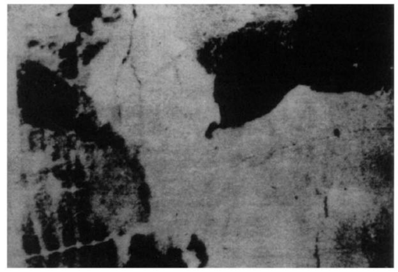
Outline

Object Representation

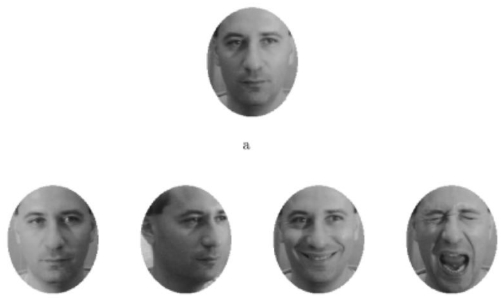
- High-level Shape Models (e.g., Generalized Cylinders)
  - Idealized images
  - Texture Less
- Mid-level Shape Models (e.g., CAD models, Superquadrics)
  - More complex
  - Well-defined geometry
- Low-level Appearance Based Models (e.g., Eigenspaces)
  - Most complex
  - Complicated shapes

A number of challenges

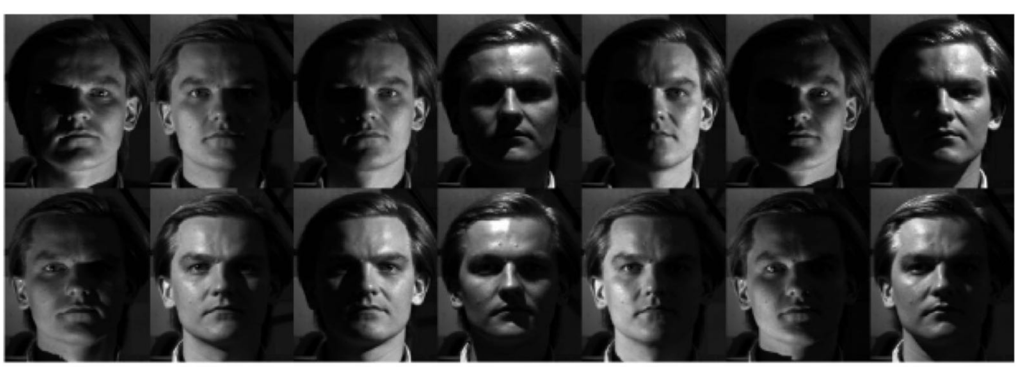
Segmentation:



Pose/Shape:



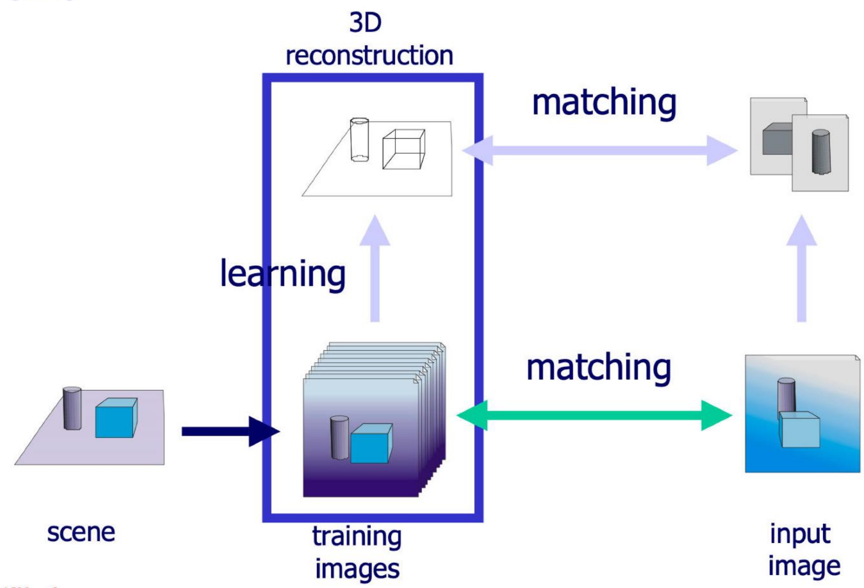
Changes in illumination



The importance of context

The importance of context - see

Learning and recognition



Outline

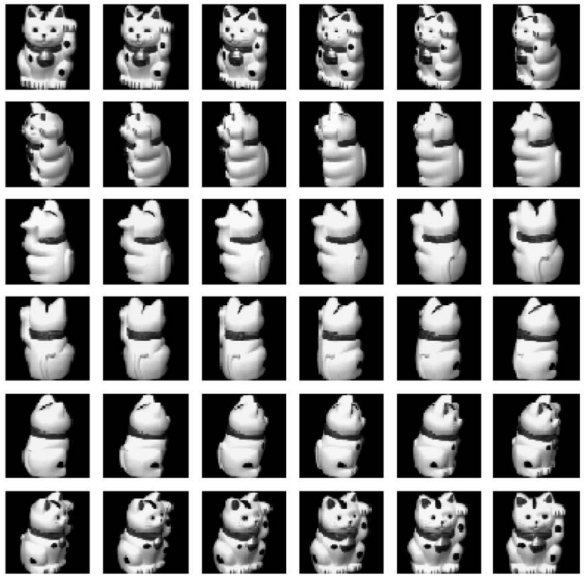
Appearance-based approaches

- The abundance of image data gives a renewed interest in appearance-based approaches
- Combined effort of:
  - Shape
  - Reflectance properties
  - Pose in the scene
  - Illumination conditions / variations
- Acquired through an automatic learning phase
- Well defined error characteristics

Numerous use-cases

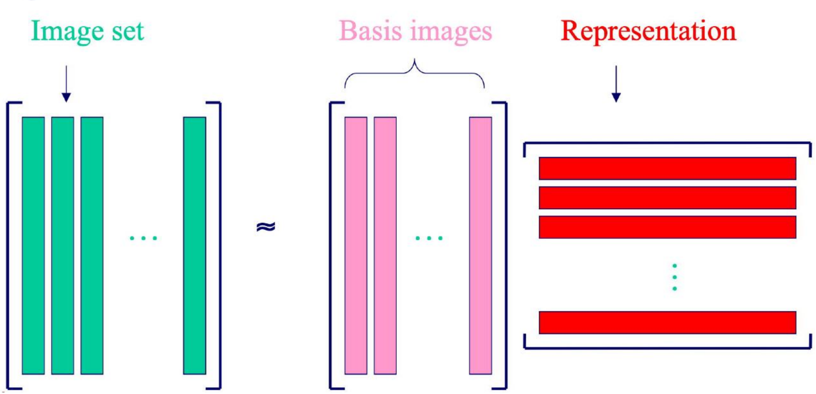
- Face-recognition (eigen faces)
- Visual inspection
- Tracking and pose estimation for robotics
- Basic object tracking
- Planning of illumination
- Image spotting
- Mobile robot localization
- ...

IDEA: Take a large number of image views



IDEA: Subspace Methods

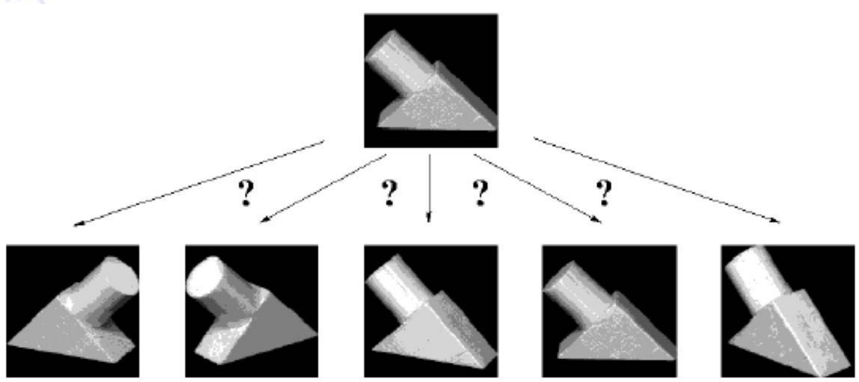
- Images are represented as points in an N-dimensional space
- Images only occupy a small fraction of the hyper-space
- Characterize the subspace / manifold spanned by the images



Multiple subspace methods

- Optimal Reconstruction ⇒ PCA
- Optimal Separation ⇒ LDA
- Optimal Correlation ⇒ CCA
- Independent Factors ⇒ ICA
- Non-negative factorization ⇒ NMF

Image matching



$$\rho = \frac{x^T y}{||x||||y||} \geq \Theta$$

Or Normalized Images

$$||x - y||^2 \leq \Phi$$

Eigenspace representation

- Image set (normalized, zero mean)

$$X = [x_0 \; x_1 \; \dots \; x_{n-1}]; \; X \in R^{m \times n}$$

- Looking for ortho-normal basis

$$U = [u_0 \; u_1 \; \dots \; u_k \;]; k \ll n$$

- Individual images are then a linear combination of basis vectors

$$x_i \approx \tilde{x}_i = \sum_{j=0}^{k} q_j(x_i) u_j$$

$$||x - y||^2 \approx || \sum_{j=0}^{k} q_j(x) u_j - \sum_{j=0}^{k} q_j(y) u_j ||^2$$

$$|| \sum_{j} q_j(x) - q_j(y) ||^2$$

Choosing a basis function?

- The optimization problem

$$\sum_{i=0}^{n-1} ||x_i - \sum_{j=0}^{k} q_j(x_i) u_j||^2 \rightarrow \min$$

- Taking k eigenvectors with the largest eigenvalues

$$C = X \; X^T = [x_0 \; x_1 \; \dots \; x_{n-1} \;] \begin{bmatrix} x_0^T \\ x_1^T \\ \dots \\ x_{n-1}^T \end{bmatrix}$$

- The PCA or Karhunen-Loeve Transform

$$C u_i = \lambda_i u_i$$

Efficient eigenspace computation

- n ≪ m
- Computing the eigenvectors u_i' i = 0, ..., n-1 of the inner product matrix

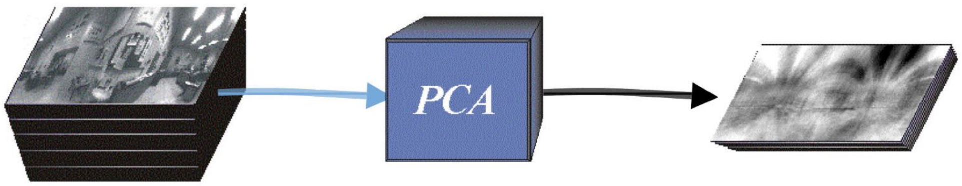
$$Q = X^T X = \begin{bmatrix} x_0^T \\ x_1^T \\ \dots \\ x_{n-1}^T \end{bmatrix} [x_0 \; x_1 \; \dots \; x_{n-1}\;]; Q \in R^{n \times n}$$

- The eigenvectors of XX^T can be obtained using $XX^T X v_i' = \lambda_i' X v_i'$:

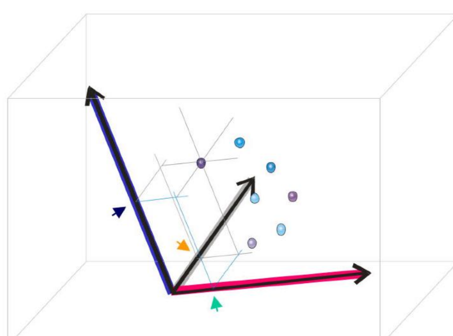
$$u_i = \frac{1}{\sqrt{\lambda_i'}} X u_i'$$

Outline

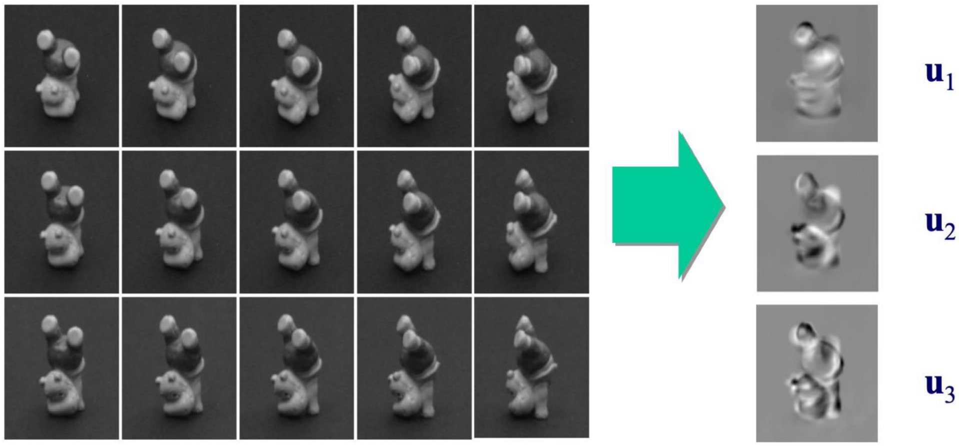
Principal Component Analysis



Principal Component Analysis



PCA Image Representation



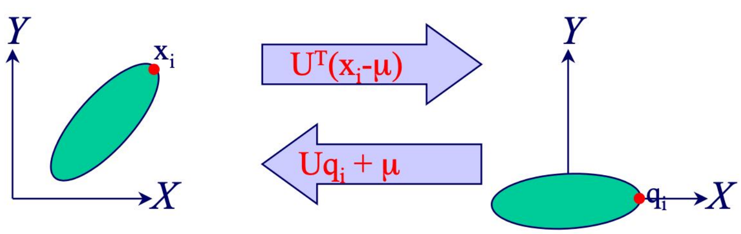
Properties of PCA

- Any point x_i can be projected to an appropriate point q_i by

$$q_i = U^T(x_i - \mu)$$

- and conversely

$$Uq_i + \mu = x_i$$



Properties of PCA

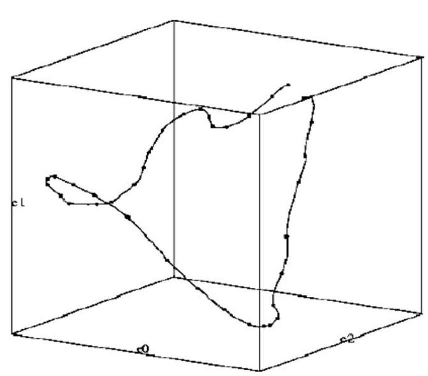
- It can be shown the MSE between x_i and its reconstruction using m eigenvectors is given by

$$\sum_{j=1}^{N} \lambda_j - \sum_{j=1}^{m} \lambda_j = \sum_{j=m+1}^{N} \lambda_j$$

- PCA minimizes the reconstruction error
- PCA maximizes the variance of projection
- Find a “natural” coordinate system for the sample data

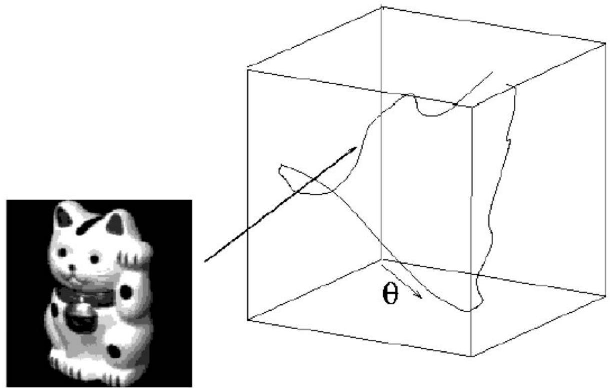
PCA for visual recognition and pose estimation

- Objects/images are represented as coordinates in an m-dimension space
- An example
  - 3D space with points representing objects on a manifold of parametric eigenspace such as orientation, pose, illumination, ...



PCA for visual recognition and pose estimation

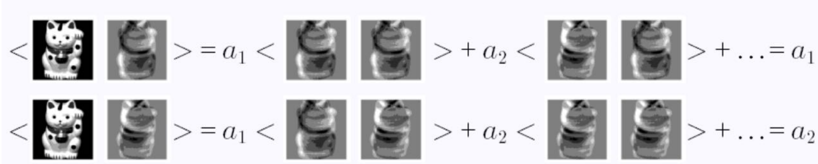
- Calculate coefficients
- Search for nearest point on manifold
- Point determines / interpolates object and/or pose



Coefficient calculation

- To recover a_i the image is projected into the eigenspace

$$a_i(\mathbf{x}) = <\mathbf{x}, \mathbf{e}_i> = \sum_{j=1}^{m} x_j e_{i_j} \qquad 1 \leq i \leq p$$

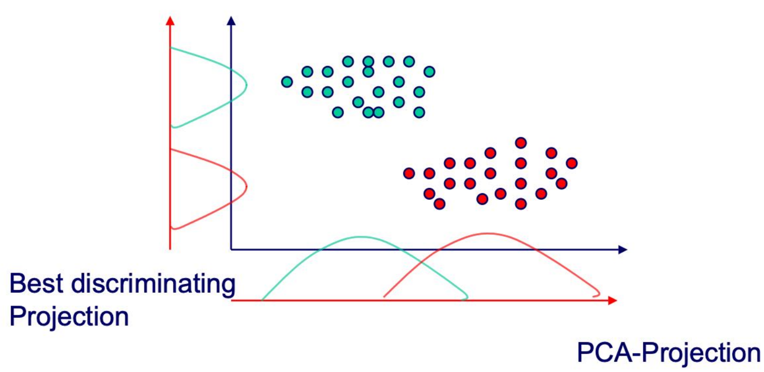


- Complete image x_i is required to calculate a_i
- Corresponds to a least square solution

Outline

Linear Discriminate Analysis

- PCA minimizes the projection error



- PCA is unsupervised – no class information is used
- Discriminating information may be used

Linear Discriminate Analysis

- For LDA would would like to
  - Maximize distance between classes
  - Minimize distance within classes
- Fisher linear discriminant

$$\rho(W) = \frac{W^T S_B W}{W^T S_W W}$$

LDA: Problem Formulation

- n sample images: $\{x_1, \ldots, x_n\}$
- c classes: $\{\chi_1, \ldots, \chi_c\}$
- Average of each class: $\mu_i = \frac{1}{n_i} \sum_{x_k \in \chi_i} x_k$
- Total average: $\mu = \frac{1}{n} \sum_{k=1}^{N} x_k$

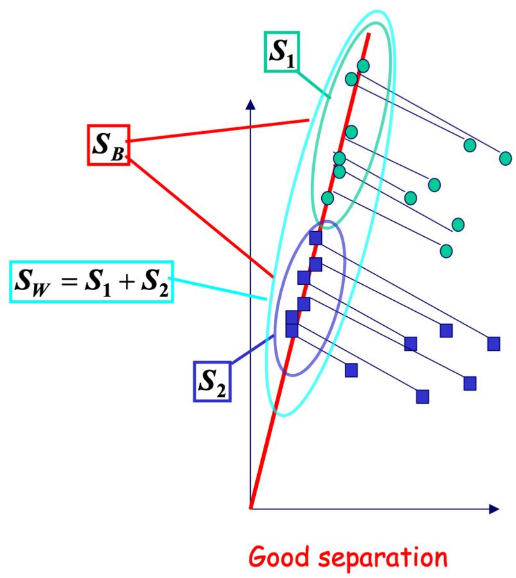
LDA: Practice

- Scatter of class i: $S_i = \sum_{x_k \in \chi_i} (x_k - \mu_i)(x_k - mu_i)^T$
- Within class scatter: $S_W = \sum_{i=1}^{c} S_i$
- Between class scatter: $S_B = \sum_{i=1}^{c} |\chi_i|(\mu_i - \mu)(\mu_i - \mu)^T$
- Total scatter: $S_T = S_W + S_B$

LDA: Practice

- After projection: $y_k = W^T x_k$
  - Between class scatter of y: $\tilde{S}_B = W^T S_B W$
  - Within class scatter of y: $\tilde{S}_W = W^T S_W W$

LDA Projection



LDA characteristics

- Maximization of
$$\rho(W) = \frac{W^T S_B W}{W^T S_W W}$$
- given by solution of generalized eigenvalue problem
$$S_B W = \lambda S_W W$$
- The the c-class case we obtain c-1 projections as the largest eigenvalue of
$$S_B W_i = \lambda S_W W_i$$

LDA in the wild

- How does one calculate LDA for high-dimensional images?
- Problem: S_W is always singular
  - Number of pixels in an image is larger than number of images in training set
- Fisherfaces example: reduce dimensionality by doing a PCA first and then LDA
- Simultaneous diagonalization of S_W and S_B

Ficherfaces

- First published by Belhumeur et al 1997
- Reduce dimensionality to n-c with PCA

$$U_{PCA} = \arg\max_U |U^T Q U| = [\ u_1\ u_2 \dots\ u_{n-c}\ ]$$

- Further reduce to c-1 with LDA

$$W_{LDA} = \arg\max_w \frac{|W^T W_{pca}^T S_B W_{pca} W|}{|W^T W_{pca}^T S_W W_{pca} W|} = [\ w_1\ w_2\ \dots\ w_{c-1}\ ]$$

- The optimal projection is then

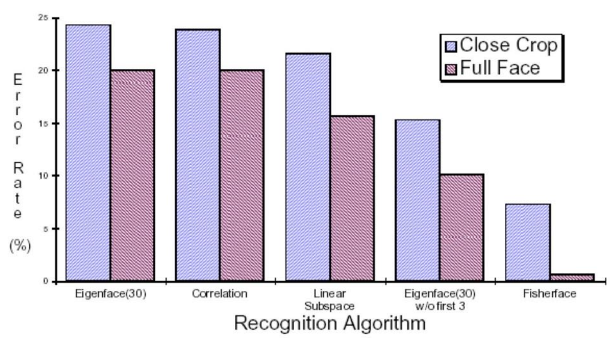
$$W_{opt} = W_{LDA}^T U^T$$

Example Fisherface

- Example Fisherface of recognition face w/wo glasses (Belhumeur et al, 1997)

Fisher example performance

- Small comparison of face recognition (old data)



- Significantly better performance than PCA for face recognition
- Noise sensitive
- Standard large scale Kaggle competitions today score 97%

Outline

Canonical Correlation Analysis (CCA)

- Also supervised method by motivated by regression / interpolation tasks such as pose estimation
- CCA related two sets of observations by determining pairs of directions that yield maximum correlation between the data sets
- Find a pair of directions (canonical factors): $w_x \in R^P$ *and* $w_y \in R^q$ so that the correlation of the projections $c = w_x^T x$ and $d = w_y^T y$ become maximal

CCA - the details

$$\rho = \frac{E[cd]}{\sqrt{E[c^2]\ E[d^2]}} =$$

$$\frac{E[w_x^T x\ y^t w_y]}{\sqrt{E[w_x^T x\ x^T w_x] E[w_y^T y\ y^t w_y]}} =$$

$$\frac{w_x^T C_{xy} w_y}{\sqrt{w_x^T C_{xx} w_x w_y^T C_{yy} w_y}}$$

CCA - computations

- Finding solutions

$$w = \begin{bmatrix} w_x \\ w_y \end{bmatrix} \quad A = \begin{bmatrix} 0 & C_{xy} \\ C_{yx} & 0 \end{bmatrix} \quad B = \begin{bmatrix} C_{xx} & 0 \\ 0 & C_{yy} \end{bmatrix}$$

- Compute the Rayleigh Quotient

$$r = \frac{w^T A w}{w^T B w}$$

- Think of it as a generalized eigenvalue problem

$$Aw = \mu B w$$

CCA for images

- Same challenge as for LDA
- Computational analysis based on SVD

$$A = C_{xx}^{-\frac{1}{2}} C_{xy} C_{yy}^{-\frac{1}{2}}$$
$$A = UDV^T$$
$$w_{xi} = C_{xx}^{-\frac{1}{2}} u_i$$
$$w_{yi} = C_{yy}^{-\frac{1}{2}} v_i$$

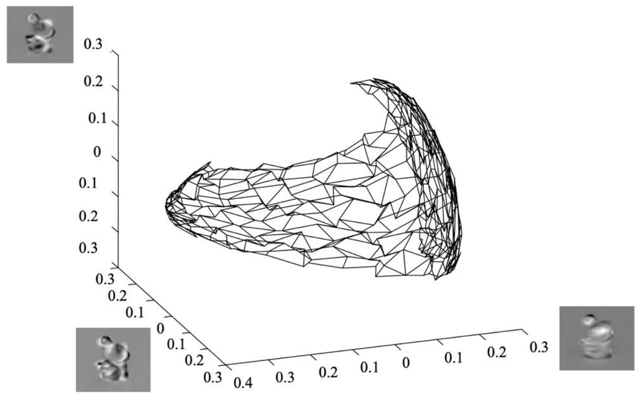
Properties of CCA

- At most min(p, q, n) CCA factors
- Invariant wrt to affine transformations
- Orthogonality of the canonical factors

$$w_{xi}^T C_{xx} w_{xj} = 0$$
$$w_{yi}^T C_{yy} w_{yj} = 0$$
$$w_{xi}^T C_{xy} w_{yj} = 0$$

CCA Example

- Parametric eigenspace obtained by PCA for 2 DOF pose space



Outline

1. Introduction
2. Appearance based learning and recognition
3. Appearance based method for visual object recognition
4. Principal Component Analysis
5. Linear Discriminative Analysis
6. Canonical Correlation Analysis
7. **Independent Component Analysis (ICA)**
8. Summary

Independent Component Analysis (ICA)

- ICA is a powerful technique from signal processing (blind source separation)
- Can we seen as an extension of PCA
- PCA takes statistics up to 2nd order into account
- ICA estimate components that are statistically independent
- Generates sparse/local descriptors - sparse coding

Independent Component Analysis (ICA)

- m scalar variables - $X = (x_1, \; \dots \; x_m)^T$
- Assumed to be a linear mixture of n sources - $S = (s_1, \; \dots \; s_n)^T$

$$X = AS$$

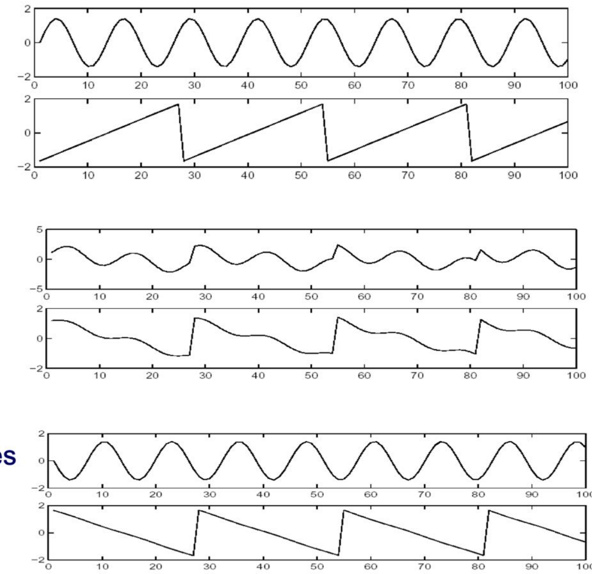
- Objective: Given X find estimates for A and S under the assumption S are independent

ICA Example

Original Sources

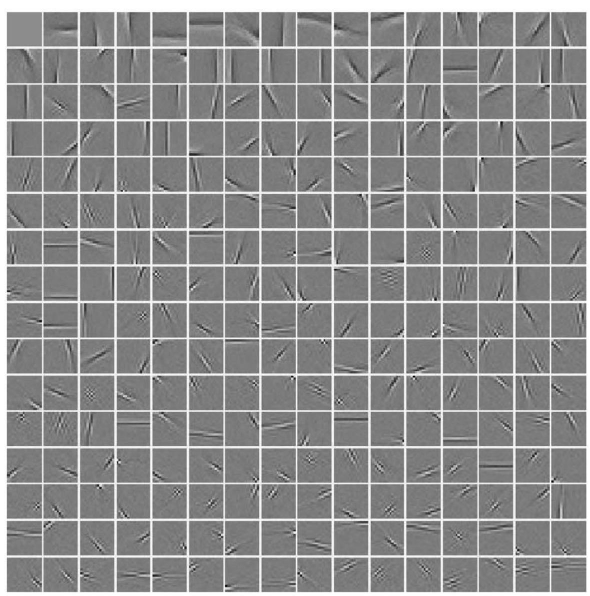
Mixtures

Recovered Sources



ICA Example

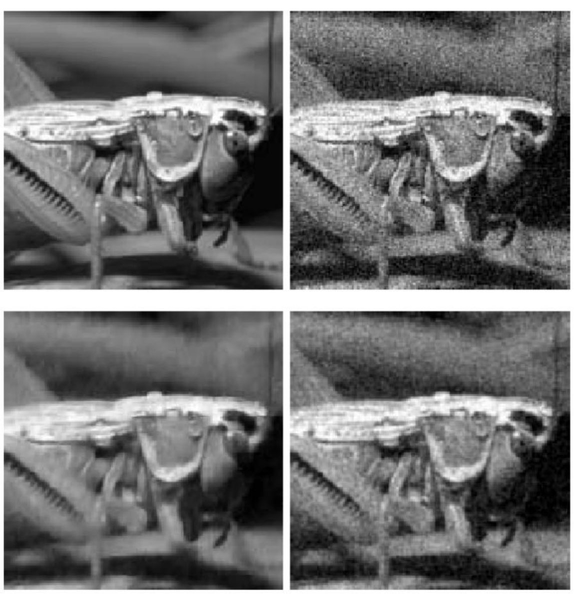
ICA basis obtained from 16x16 patches of natural images (Bell&Sejnowski 96)



ICA Algorithms

- Minimize a complex tensor function
- Adaptive algorithms based on stochastic gradient
  - Measure independence
  - Computer A recursively to maximize independence
- ICA only works for non-Gaussian sources
- Often whitening of data is performance
- ICA does not provide ordering
- ICA components are not orthogonal

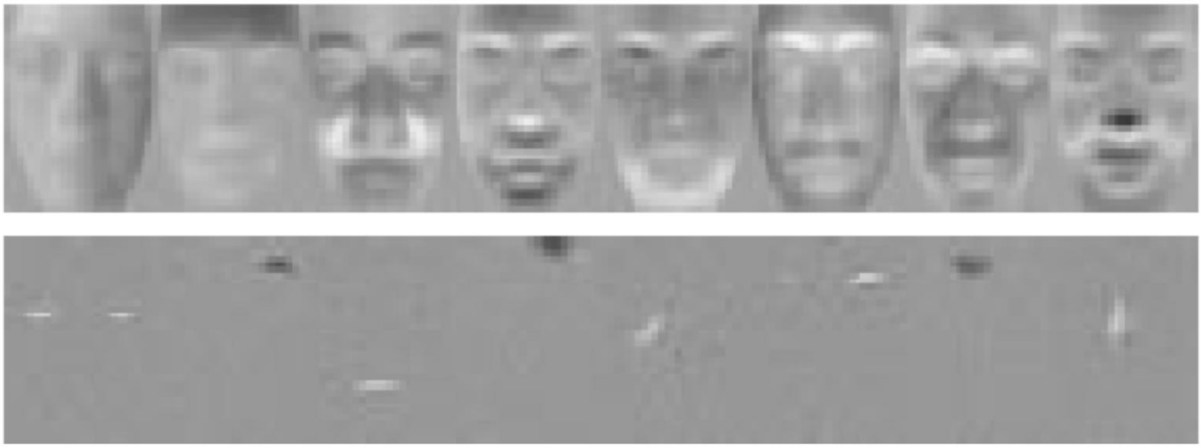
ICA noise suppression example



Example from Hyvärinen, 1999

PCA vs ICA for face recognition

PCA



ICA

From Baek et al, 2002

Outline

Summary

- Brief overview of use of sub-space methods for data processing
- The exact task should dictate the choice of methods
- Other cascaded processing simplifies complexity
- Good standard tools available in most signal processing toolboxes

Questions

Questions