CSE276C - Mathematics for Robotics

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Introduction

- Lecturer
- Structure
- Materials
- Information Sources
- Transformations

Henrik I Christensen

- Professor at UCSD
- Director of Robotics
- "Real Systems for Real Problems"
- Multi-Robot coordination
- Autonomous Driving Vehicles
- First commercial robot vacuum cleaner
- Working with Amazon, Boeing, Nissan, Qualcomm, Robust.AI, ...

Teaching Assistants

- Andi Frank, PhD student CSE - Affordance based robotics
- Office hours to be fully finalized

Structure of course

- Lectures
- Homework
- Discussions

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- Discussions

- Linear Systems
- Subspace Methods
- Optimization
- Root Finding
- Integration
- Differential Geometry
- Space & Search

Lectures

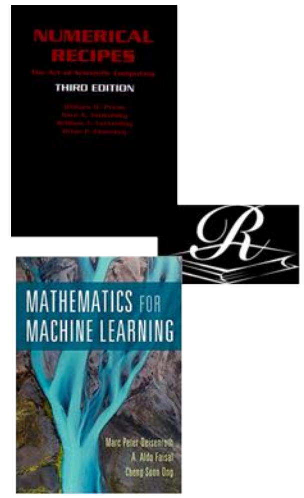
- Slides available by lecture time (PDF)
- Lectures (audio) recorded and available within 24 hours on Canvas
- Any and all feedback on format, ... is most welcome

Objectives

- Basic tools for study of robotics
- Core mathematical concepts
- A few example applications
- What are key tools for perception, planning, and basic control

Textbooks

- W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling. “Numerical Recipes”. Cambridge University Press. (Any edition.)
- T. Bewley, Numerical Renaissance: simulation, optimization, & control
- M. Deisenroth, A. Aldo Faisal, and C. Soon Ong, “Mathematics for Machine Learning”, Cambridge University Press, 2019



Information Sources

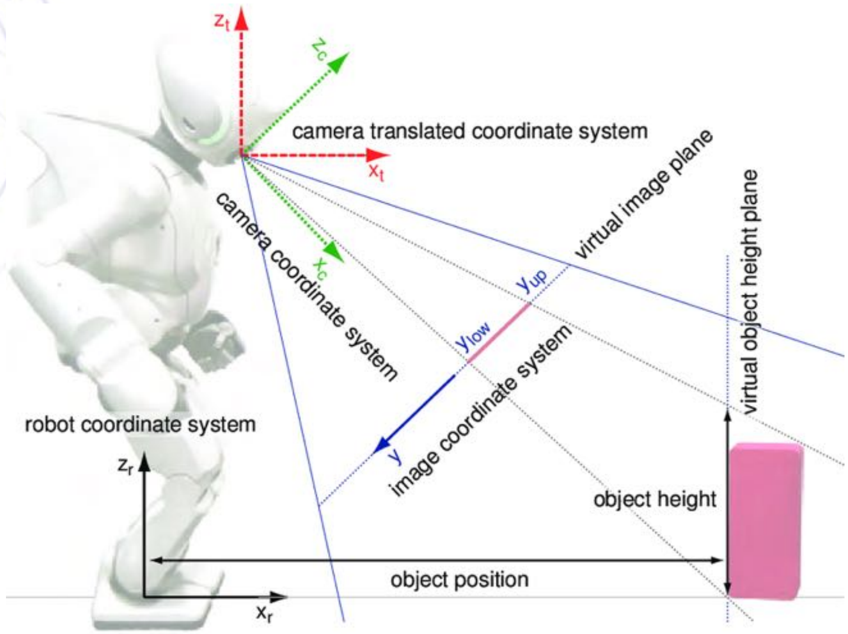
- CANVAS website
- WebSite - http://www.hichristensen.com/CSE276C-23
    - Slides
    - Lecture notes (handwritten sorry!)
- Piazza - Did you all get an invite?
- Office Hours - Henrik & TA/Andi
- Audio/podcast recordings available from CANVAS site

Homework

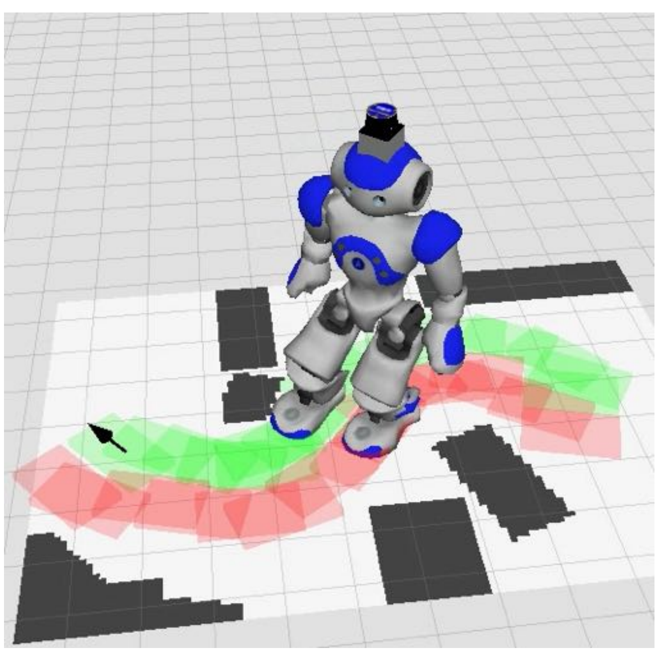
- 5 Homework assignments $\approx$ two weeks
- Some basic math - analysis by manual or automated
- Math problems in robotics (could be Python/Numpy or MatLab)
- Analysis of sample robotics data

QUESTIONS?

Use of Math in Robotics?



Use of Math in Robotics?



Space and Rotations

- How do you represent the position of a robot in space?

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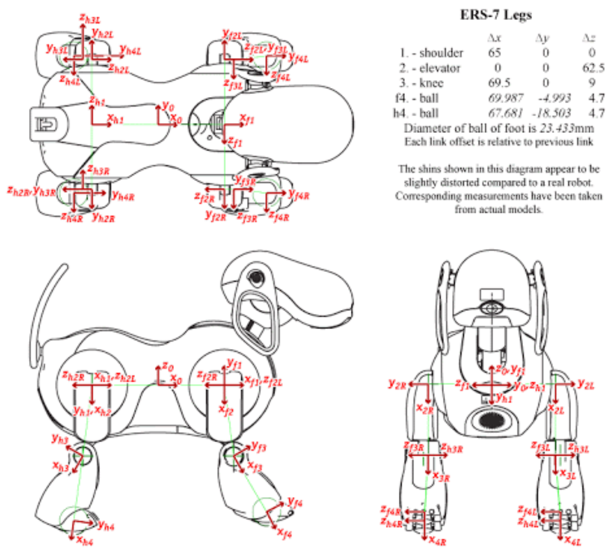
$$
{}^{j}p_{i} = \begin{pmatrix} {}^{j}p_{x_i} \\ {}^{j}p_{y_i} \\ {}^{j}p_{z_i} \end{pmatrix}
$$

- The position of i with respect to j
- examples
  - World reference frame
  - Position of the robot
  - Sensor position or a sensor point

Example Reference Frames



Rotation between two reference frames - i, j

$$
{}^{j}\mathbf{R}_i = \begin{pmatrix} \vec{x}_i\vec{x}_j & \vec{y}_i\vec{x}_j & \vec{z}_i\vec{x}_j \\ \vec{x}_i\vec{y}_j & \vec{y}_i\vec{y}_j & \vec{z}_i\vec{y}_j \\ \vec{x}_i\vec{z}_j & \vec{y}_i\vec{z}_j & \vec{z}_i\vec{z}_j \end{pmatrix}
$$

where $(\vec{x}_i, \vec{y}_i, \vec{z}_i)$ and $(\vec{x}_j, \vec{y}_j, \vec{z}_j)$ are basis vectors for the two coordinate frames

Elementary Rotations

- Rotation around Z-axis

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- the same for Y and X

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Considerations for rotations

- We can do combinations

$$^{k}\mathbf{R}_i = {}^{k}\mathbf{R}_j \, {}^{j}\mathbf{R}_i$$

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- We can do combinations

$$^{k}\mathbf{R}_i = {}^{k}\mathbf{R}_j \, {}^{j}\mathbf{R}_i$$

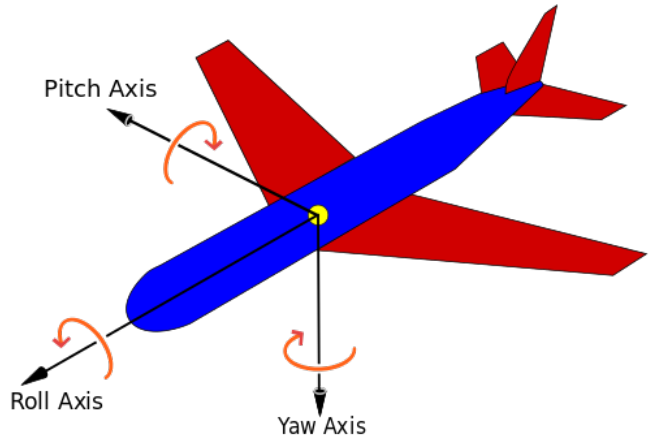
- Note the order is important

$$^{k}\mathbf{R}_j \, {}^{j}\mathbf{R}_i \neq {}^{j}\mathbf{R}_i \, {}^{k}\mathbf{R}_j$$

- The order and reference frames are very important

Euler Angles

- We frequently use Euler angles in robotics



Euler Angles

- The convention used is $R_z R_y R_x$ with respect to $(\alpha, \beta, \gamma)^T$

$$
{}^j\mathbf{R}_i = \begin{pmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{pmatrix}
$$

Derivation of Euler angles

- If we have the rotation matrix

$$ {}^{j}\mathbf{R}_{i} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{23} & r_{33} \end{pmatrix} $$

- derivation of the Euler Angles

$$ \begin{aligned} \beta &= atan2 \frac{-r_{31}}{\sqrt{r_{11}^2 + r_{21}^2}} \\ \alpha &= atan2 \frac{r_{21}/\cos\beta}{r_{11}/\cos\beta} \\ \gamma &= atan2 \frac{r_{32}/\cos\beta}{r_{33}/\cos\beta} \end{aligned} $$

When may we have problems?

- Could we have problems? / When?

When may we have problems?

- Could we have problems? / When?
- What about singularities?

QUESTIONS?

Quaternions

- Could we generate a representation that has no singularities?
- Hamilton, 1843.
- A 3-parameter family is not adequate (proved by now)
- A 4-parameter model is a possibility
- Quaternions is a possible representation (not the most intuitive)

Quaternions

- Imagine 3-D imaginary numbers - three basis vectors - $\vec{i}, \vec{j}, \vec{k}$
- We can represent a quaternion as

$$\vec{\epsilon} = \epsilon_0 + \epsilon_1 \vec{i} + \epsilon_2 \vec{j} + \epsilon_3 \vec{k}$$

or $(\epsilon_0, \vec{\epsilon})$

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- we have three basis vectors

$$\vec{i}\vec{i} = \vec{j}\vec{j} = \vec{k}\vec{k} = -1$$

- mixed products

$$\vec{i}\vec{j} = \vec{k},\ \vec{j}\vec{k} = \vec{i},\ \vec{k}\vec{i} = \vec{j}$$

$$\vec{j}\vec{i} = -\vec{k},\ \vec{k}\vec{j} = -\vec{i},\ \vec{i}\vec{k} = -\vec{j}$$

- Null quaternion

$$\vec{0} = 0 + 0\vec{i} + 0\vec{j} + 0\vec{k}$$

- Unit quarternion

$$\vec{1} = 1 + 0\vec{i} + 0\vec{j} + 0\vec{k}$$

Quaternion operations

- Product of two quaternions

$$\begin{aligned} \vec{a}\vec{b} \quad = & \quad a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 \\ + & \quad (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2)\vec{i} \\ + & \quad (a_0 b_2 + a_2 b_0 + a_3 b_1 - a_1 b_3)\vec{j} \\ + & \quad (a_0 b_3 + a_3 b_0 + a_1 b_2 - a_2 b_1)\vec{k} \end{aligned}$$

- The good news there are standard libraries

Rotations w. Quaternions

- Rotating at an angle θ around the vector $\vec{a}$ expressed as a quaternion:

$$\vec{\epsilon} = \cos\frac{\theta}{2} + a_x \sin\frac{\theta}{2}\vec{i} + a_y \sin\frac{\theta}{2}\vec{j} + a_z \sin\frac{\theta}{2}\vec{k}$$

- or

$$\vec{\epsilon} = (\cos\frac{\theta}{2}, \vec{a}\sin\frac{\theta}{2})$$

Mapping quaternions to rotation matrices

- The rotation of a quaternion $\vec{\epsilon}$ can be written as

$$^{j}\mathbf{R}_i = \begin{pmatrix} 1-2(\epsilon_2^2+\epsilon_3^2) & 2(\epsilon_1\epsilon_2-\epsilon_0\epsilon_3) & 2(\epsilon_1\epsilon_3+\epsilon_0\epsilon_2) \\ 2(\epsilon_1\epsilon_2+\epsilon_0\epsilon_3) & 1-2(\epsilon_1^2+\epsilon_3^2) & 2(\epsilon_2\epsilon_3-\epsilon_0\epsilon_1) \\ 2(\epsilon_1\epsilon_3-\epsilon_0\epsilon_2) & 2(\epsilon_2\epsilon_3+\epsilon_0\epsilon_1) & 1-2(\epsilon_1^2+\epsilon_2^2) \end{pmatrix}$$

- As I said the good news there are standard libraries

From a rotation matrix to quaternions

- Direct computing the quaternions from a rotation matrix ($\mathbf{R}$)

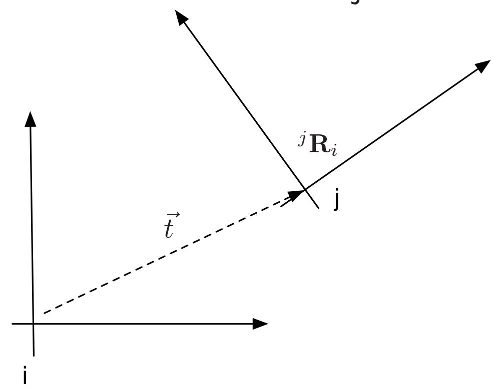
$$
\begin{aligned}
\epsilon_0 &= \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}} \\
\epsilon_1 &= \frac{r_{32} - r_{33}}{4\epsilon_0} \\
\epsilon_2 &= \frac{r_{13} - r_{31}}{4\epsilon_0} \\
\epsilon_3 &= \frac{r_{21} - r_{12}}{4\epsilon_0}
\end{aligned}
$$

- Quaternions frequently used in graphics, computer vision and robotics

QUESTIONS?

Coordinate transformations

- To move between transformation or move an object we frequently encounter



- We can write this as ${}^j\vec{p} = {}^j\mathbf{R}_i \, {}^i\vec{p} + \vec{t}$

Homogeneous Transformations

- We can do this more easily with homogeneous coordinates

$$\vec{P} = \begin{pmatrix} \vec{p} \\ 1 \end{pmatrix}$$

- given this our transformation can now be written as

$$\begin{pmatrix} {}^jp \\ 1 \end{pmatrix} = \begin{pmatrix} {}^j\mathbf{R}_i & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} {}^ip \\ 1 \end{pmatrix}$$

- or ${}^jP = {}^jT_i \, {}^iP$

Standard Joints

- Revolute joint

$$
{}^{j}\mathbf{R}_i = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

- Prismatic joint

$$
T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} & 0 \\ I & 0 \\ & d \\ 0 & 1 \end{pmatrix}
$$

Standard Joints (cont.)

- Cylindrical

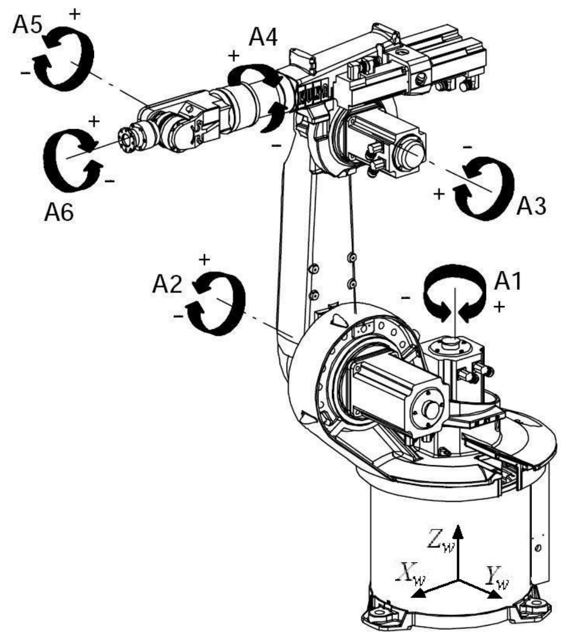
$$
T = \begin{pmatrix} & 0 \\ R_\theta & 0 \\ & d \\ 0 & 1 \end{pmatrix}
$$

- Spherical

$$
T = \begin{pmatrix} R_{\alpha,\beta,\gamma} & 0 \\ 0 & 1 \end{pmatrix}
$$

- Most other joints can be constructed from these basic models

Example Robot KUKA KR15



Robot dynamics

- This is an entire field of its own.
- How can we computer the velocity of a robot?
- If we know a desired velocity, how fast should we turn the wheels?
- In general we will refer to the robot actuators as q_i
- Forward kinematics
$$v = \Phi \dot{q}$$
- Inverse kinematics
$$\dot{q} = \Phi^{-1} v$$
- Not get into the much of the details until we talk about differential geometry (end of course)

Wrap-up

- Basic course information
  - Books, topics, websites, ...
- Introduction to positions, rotations, transformations, ...
- Next time we will talk about linear systems of equations

Questions

Questions